

On perturbative limits of quadrupole evolution in QCD at high energy

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We consider the perturbative (weak field) limit of the small x QCD evolution equation for quadrupole, the normalized trace of four Wilson lines in the fundamental representation, which appears in di-hadron angular correlation in high energy collisions. We linearize the quadrupole evolution equation and then expand the Wilson lines in powers of $g A_\mu$ where A_μ is the gauge field. The quadratic terms in the expansion ($\sim g^2 A^2$) satisfy the BFKL equation as has been recently shown. We then consider the quartic terms ($\sim g^4 A^4$) in the expansion and show that the linearized quadrupole evolution equation, written in terms of color charge density ρ , reduces to the well-known BJKP equation for the imaginary part of four-reggeized gluon exchange amplitude. We comment on the possibility that the BJKP equation for the evolution of a n -reggeized gluon state can be obtained from the JIMWLK evolution equation for the normalized trace of n fundamental Wilson lines when non-linear (recombination) terms are neglected.

I. INTRODUCTION

The recent experimental observation of disappearance of the away side peak in di-hadron angular correlation in the forward rapidity region in deuteron-gold collisions at RHIC [1] has generated a lot of interest in multi-parton correlations at high energy (small x). Unlike structure functions in DIS and single inclusive particle production in hadronic collisions which are sensitive to dipoles (normalized trace of two Wilson lines), di-hadron correlations probe correlators of higher number of Wilson lines [2, 3]. Therefore one has the opportunity, for the first time, to investigate these higher correlators experimentally through studies of angular and rapidity correlations in di-hadron production cross section in high energy hadronic collisions. Such studies can teach us much about the intrinsic correlations in the hadronic or nuclear wave functions which are not accessible in single inclusive particle production or in studies of structure function in DIS.

Higher correlators of Wilson lines appear in two-hadron production cross section in any dilute-dense collision at high energy where analytic calculations are possible. Classic examples of such asymmetric collisions are proton-nucleus collisions (see [4] for a review) in the fragmentation region of the proton, and in DIS close to the virtual photon remnants¹. Two-gluon production cross section in DIS has been considered in [2] while two-parton production cross section in proton-nucleus collisions has been investigated in [2, 5–8]. In all cases, the cross section involves correlators of higher (more than two) number of Wilson lines, the most important being the quadrupole operator. Evolution equations for these higher point correlators have been derived [2, 9, 10] and approximate analytic expressions for them have been developed using a Gaussian model [11] and approximate analytic solutions have been proposed [12]. Very recently, powerful lattice gauge theory techniques have been applied to solve the JIMWLK evolution equation which then allows a systematic and detailed numerical study of the properties of these higher point correlators [13].

Here we study the evolution equation for the quadrupole operator in the weak field limit. A first study of this has already been performed in [10] where it is shown that the quadrupole evolution equation reduces to a sum of BFKL equations for the dipole operator in the limit where the dipole is expanded in powers of the gluon field and quadratic terms in gluon field are kept. Here, we go beyond the quadratic expansion and show that the quartic terms in the expansion of the linearized quadrupole evolution equation satisfy an equation which is identical to the BJKP equation [14, 15] for the imaginary part of the four-reggeized gluon exchange amplitude. This should be very useful since there is an extensive literature on the properties of the BJKP equation which may give us further insight on the properties of the JIMWLK equation in the limit where one may ignore non-linear terms.

¹ Particle production in the very forward rapidity region in proton-proton collisions at very high energy falls into this category also.

II. QUADRUPOLE EVOLUTION EQUATION

We start by defining the quadrupole operator Q as

$$Q(r, \bar{r}, \bar{s}, s) \equiv \frac{1}{N_c} \text{tr} V_r V_{\bar{r}}^\dagger V_{\bar{s}} V_s^\dagger \quad (1)$$

where $V_r \equiv V(r_t)$ is a Wilson line in the fundamental representation in the covariant gauge

$$V(r_t) \equiv \hat{P} e^{-ig \int dx^- A^+} \quad (2)$$

and $A^\mu(x^-, r_t) = \delta^\mu + \delta(x^-) \alpha(r_t)$. The gauge field $\alpha(r_t)$ is related to the color charge density via $\partial_t^2 \alpha^a(r_t) \sim g \rho^a(r_t)$ and r, \bar{r}, \bar{s}, s etc. denote two-dimensional coordinates on the transverse plane. The evolution equation for the quadrupole was derived in [2] in the large N_c limit and using Feynman diagram techniques. It has been recently re-derived [10] using the JIMWLK equation where it was shown that there are no N_c suppressed corrections. Here we outline the derivation using the JIMWLK formalism [16] where the evolution ($y = \log 1/x$) of any operator is given by

$$\frac{d}{dy} \langle O \rangle = \frac{1}{2} \left\langle \int d^2x d^2y \frac{\delta}{\delta \alpha_x^b} \eta_{xy}^{bd} \frac{\delta}{\delta \alpha_y^d} O \right\rangle, \quad (3)$$

with

$$\eta_{xy}^{bd} = \frac{1}{\pi} \int \frac{d^2z}{(2\pi)^2} \frac{(x-z) \cdot (y-z)}{(x-z)^2 (y-z)^2} [1 + U_x^\dagger U_y - U_x^\dagger U_z - U_z^\dagger U_y]^{bd}. \quad (4)$$

and U is a Wilson line in the adjoint representation. The derivation of the quadrupole evolution equation is straightforward but tedious. It involves functional differentiation of the Wilson lines and repeated use of the identity $[U(r)]^{ab} t^b = V^\dagger(r) t^a V(r)$. The result is

$$\begin{aligned} \frac{d}{dy} \langle Q(r, \bar{r}, \bar{s}, s) \rangle = & \frac{N_c \alpha_s}{(2\pi)^2} \int d^2z \left\{ \left\langle \left[\frac{(r-\bar{r})^2}{(r-z)^2 (\bar{r}-z)^2} + \frac{(r-s)^2}{(r-z)^2 (s-z)^2} - \frac{(\bar{r}-s)^2}{(\bar{r}-z)^2 (s-z)^2} \right] Q(z, \bar{r}, \bar{s}, s) S(r, z) \right. \right. \\ & + \left[\frac{(r-\bar{r})^2}{(r-z)^2 (\bar{r}-z)^2} + \frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2 (\bar{s}-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2 (\bar{s}-z)^2} \right] Q(r, z, \bar{s}, s) S(z, \bar{r}) \\ & + \left[\frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2 (\bar{s}-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2 (\bar{s}-z)^2} - \frac{(\bar{r}-s)^2}{(s-z)^2 (\bar{r}-z)^2} \right] Q(r, \bar{r}, z, s) S(\bar{s}, z) \\ & + \left[\frac{(r-s)^2}{(r-z)^2 (s-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2 (\bar{s}-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2 (\bar{s}-z)^2} \right] Q(r, \bar{r}, \bar{s}, z) S(z, s) \\ & - \left[\frac{(r-\bar{r})^2}{(r-z)^2 (\bar{r}-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2 (\bar{s}-z)^2} + \frac{(r-s)^2}{(r-z)^2 (s-z)^2} + \frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2 (\bar{s}-z)^2} \right] Q(r, \bar{r}, \bar{s}, s) \\ & - \left[\frac{(r-s)^2}{(r-z)^2 (s-z)^2} + \frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2 (\bar{s}-z)^2} - \frac{(\bar{r}-s)^2}{(\bar{r}-z)^2 (s-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2 (\bar{s}-z)^2} \right] S(r, s) S(\bar{r}, \bar{s}) \\ & \left. - \left[\frac{(r-\bar{r})^2}{(r-z)^2 (\bar{r}-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2 (\bar{s}-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2 (\bar{s}-z)^2} - \frac{(\bar{r}-s)^2}{(\bar{r}-z)^2 (s-z)^2} \right] S(r, \bar{r}) S(\bar{s}, s) \right\} \quad (5) \end{aligned}$$

where the S matrix is defined as

$$S(r, \bar{r}) \equiv \frac{1}{N_c} \text{tr} V_r V_{\bar{r}}^\dagger \quad (6)$$

We will refer to the first four lines in this equation as "real" and the last three terms as "virtual" terms in coordinate space. This is to distinguish them from the real and virtual terms in momentum space after we Fourier transform the equation since there is no one to one correspondence between the real and virtual terms in coordinate and momentum spaces. We have also verified that this equation is exact in the sense that there are no N_c suppressed terms in the equation itself (note that models used to evaluate the color averaging denoted by $\langle \dots \rangle$ may introduce sub-leading

N_c terms). It also agrees with the previous results for the quadrupole evolution equation [2, 10]. The S matrix satisfies the BK evolution equation [17] given by

$$\frac{d}{dy} \langle S(r-s) \rangle = \frac{N_c \alpha_s}{2\pi^2} \int d^2z \frac{(r-s)^2}{(r-z)^2(s-z)^2} \left[\langle S(r-z) \rangle \langle S(z-s) \rangle - \langle S(r-s) \rangle \right] \quad (7)$$

Unlike the dipole kernel in the BK equation which allows a probabilistic interpretation in coordinate space, the same is not true in the quadrupole evolution equation due to terms with negative signs. Even though the individual kernels in eq. (5) are just the standard dipole kernels [18], it is still perhaps useful to explain in a more intuitive way, the various terms that appear in eq. (5). The first four lines in eq. (5) are the "real" corrections and come from the third and fourth terms in eq. (4). One can rewrite any kernel in eq. (5) in a way which may look more familiar and facilitates the comparison with the standard dipole emission kernel. For example, the kernel in the first line on the right hand side of eq. (5) can be written as as

$$\sim 2 \left[\frac{1}{(r-z)^2} - \frac{(r-z) \cdot (\bar{r}-z)}{(r-z)^2(\bar{r}-z)^2} - \frac{(r-z) \cdot (s-z)}{(r-z)^2(s-z)^2} + \frac{(\bar{r}-z) \cdot (s-z)}{(\bar{r}-z)^2(s-z)^2} \right] \quad (8)$$

with a similar form for all the other kernels. Here the first term corresponds to a gluon being radiated by a quark line represented by $V(r)$. If it is absorbed by the same quark line in the amplitude, it leaves the quadrupole unchanged and will correspond to a "virtual" correction. On the other hand if it is absorbed by the same quark line but in the complex conjugate amplitude (so the gluon line crosses the cut), it will multiply the quadrupole with the coordinate r replaced by z and a dipole with coordinates r, z . This will be part of the "real" corrections. The second term above corresponds to the case when the quark line, represented by $V(r)$, in the quark anti-quark system represented by $V(r)$ and $V(\bar{r})$ radiates a gluon with transverse coordinate z . If the radiated gluon does not cross the cut line and is absorbed by the anti-quark line at \bar{r} it becomes part of the "virtual" corrections. On the other hand if the radiated gluon at z crosses the cut and is then absorbed by an anti-quark line in the complex conjugate amplitude, it breaks the original quadrupole into a quadrupole with coordinate r replaced by z and a dipole at r, z . This is part of the "real" corrections. All other terms have a similar interpretation.

To investigate the weak field limit of this evolution and to make our approximations more transparent, it is more useful to work with the T matrices, defined as $T_Q \equiv 1 - Q$ and $T \equiv 1 - S$. It is easy to see that all kernels multiplying 1 (when we switch from Q, S to T_Q, T) add up to zero. Therefore, eq. (5) is re-written as

$$\begin{aligned} \frac{d}{dy} \langle T_Q(r, \bar{r}, \bar{s}, s) \rangle &= \frac{N_c \alpha_s}{(2\pi)^2} \int d^2z \left\{ \left[\frac{(r-\bar{r})^2}{(r-z)^2(\bar{r}-z)^2} + \frac{(r-s)^2}{(r-z)^2(s-z)^2} - \frac{(\bar{r}-s)^2}{(\bar{r}-z)^2(s-z)^2} \right] [T_Q(z, \bar{r}, \bar{s}, s) + T(r, z) - T_Q(z, \bar{r}, \bar{s}, s)T(r, z)] \right. \\ &+ \left[\frac{(r-\bar{r})^2}{(r-z)^2(\bar{r}-z)^2} + \frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2(\bar{s}-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2(\bar{s}-z)^2} \right] [T_Q(r, z, \bar{s}, s) + T(z, \bar{r}) - T_Q(r, z, \bar{s}, s)T(z, \bar{r})] \\ &+ \left[\frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2(\bar{s}-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2(\bar{s}-z)^2} - \frac{(s-\bar{r})^2}{(s-z)^2(\bar{r}-z)^2} \right] [T_Q(r, \bar{r}, z, s) + T(\bar{s}, z) - T_Q(r, \bar{r}, z, s)T(\bar{s}, z)] \\ &+ \left[\frac{(r-s)^2}{(r-z)^2(s-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2(\bar{s}-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2(\bar{s}-z)^2} \right] [T_Q(r, \bar{r}, \bar{s}, z) + T(z, s) - T_Q(r, \bar{r}, \bar{s}, z)T(z, s)] \\ &- \left[\frac{(r-\bar{r})^2}{(r-z)^2(\bar{r}-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2(\bar{s}-z)^2} + \frac{(r-s)^2}{(r-z)^2(s-z)^2} + \frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2(\bar{s}-z)^2} \right] T_Q(r, \bar{r}, \bar{s}, s) \\ &- \left[\frac{(r-s)^2}{(r-z)^2(s-z)^2} + \frac{(\bar{r}-\bar{s})^2}{(\bar{r}-z)^2(\bar{s}-z)^2} - \frac{(\bar{r}-s)^2}{(\bar{r}-z)^2(s-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2(\bar{s}-z)^2} \right] [T(r, s) + T(\bar{r}, \bar{s}) - T(r, s)T(\bar{r}, \bar{s})] \\ &- \left. \left[\frac{(r-\bar{r})^2}{(r-z)^2(\bar{r}-z)^2} + \frac{(s-\bar{s})^2}{(s-z)^2(\bar{s}-z)^2} - \frac{(r-\bar{s})^2}{(r-z)^2(\bar{s}-z)^2} - \frac{(\bar{r}-s)^2}{(\bar{r}-z)^2(s-z)^2} \right] [T(r, \bar{r}) + T(\bar{s}, s) - T(r, \bar{r})T(\bar{s}, s)] \right\} \quad (9) \end{aligned}$$

A. The weak field limits

It is useful to consider the above equation for T_Q in the weak field (dilute) limit where all sizes are much smaller than the inverse saturation scale, i.e., $|a-b| \ll \frac{1}{Q_s}$ for any external coordinates a, b . In this limit the non-linear

terms ($T_Q T$ and TT) in eq. (9) may be dropped and we get

$$\begin{aligned} \frac{d}{dy} \langle T_Q(r, \bar{r}, \bar{s}, s) \rangle = \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 z \left\{ \left[\frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(r - s)^2}{(r - z)^2 (s - z)^2} - \frac{(\bar{r} - s)^2}{(\bar{r} - z)^2 (s - z)^2} \right] [T_Q(z, \bar{r}, \bar{s}, s) + T(r, z)] + \right. \\ \left[\frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} \right] [T_Q(r, z, \bar{s}, s) + T(z, \bar{r})] + \\ \left[\frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} - \frac{(s - \bar{r})^2}{(s - z)^2 (\bar{r} - z)^2} \right] [T_Q(r, \bar{r}, z, s) + T(\bar{s}, z)] + \\ \left[\frac{(r - s)^2}{(r - z)^2 (s - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} \right] [T_Q(r, \bar{r}, \bar{s}, z) + T(z, s)] - \\ \left[\frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} + \frac{(r - s)^2}{(r - z)^2 (s - z)^2} + \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} \right] T_Q(r, \bar{r}, \bar{s}, s) - \\ \left[\frac{(r - s)^2}{(r - z)^2 (s - z)^2} + \frac{(\bar{r} - \bar{s})^2}{(\bar{r} - z)^2 (\bar{s} - z)^2} - \frac{(\bar{r} - s)^2}{(\bar{r} - z)^2 (s - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} \right] [T(r, s) + T(\bar{r}, \bar{s})] - \\ \left. \left[\frac{(r - \bar{r})^2}{(r - z)^2 (\bar{r} - z)^2} + \frac{(s - \bar{s})^2}{(s - z)^2 (\bar{s} - z)^2} - \frac{(r - \bar{s})^2}{(r - z)^2 (\bar{s} - z)^2} - \frac{(\bar{r} - s)^2}{(\bar{r} - z)^2 (s - z)^2} \right] [T(r, \bar{r}) + T(\bar{s}, s)] \right\} \end{aligned} \quad (10)$$

To proceed further, we first consider the two-gluon exchange limit, i.e., the BFKL equation [19]. Since T_Q and T include multiple gluon exchanges, we need to linearize them, i.e., take the single (reggeized) gluon exchange limit. This corresponds to expanding each of the Wilson lines in the definition of T_Q and T to first order in the gauge field α and then keeping terms of the order α^2 . In this limit (note the relative sign which appears when taking both α 's from either V 's or V^\dagger 's rather than taking one α from a V and another α from a V^\dagger)

$$T_Q(r, \bar{r}, \bar{s}, s) \rightarrow T(r, \bar{r}) + T(\bar{s}, s) - T(r, \bar{s}) - T(\bar{r}, s) + T(r, s) + T(\bar{r}, \bar{s}) \quad (11)$$

Using eq. (11) in both sides of eq. (10) we get the BFKL equation for each T of a given argument. For example,

$$\frac{d}{dy} \langle T(r, s) \rangle = \frac{N_c \alpha_s}{2\pi^2} \int d^2 z \frac{(r - s)^2}{(r - z)^2 (s - z)^2} \left[\langle T(r, z) \rangle + \langle T(z, s) \rangle - \langle T(r, s) \rangle \right] \quad (12)$$

where T in eq. (12) and right hand side of (11) stands for

$$T(r, \bar{r}) \rightarrow \Gamma(r - \bar{r}) \sim g^2 \alpha^a(r) \alpha^a(\bar{r}) \quad (13)$$

This limit was already considered in [10] and the correspondence with BFKL was shown. We also note that this relation still holds when the evolution equation is written in terms of the color charge density ρ rather than the gauge field α .

The next interesting case is to consider $O(\alpha^4)$ and see whether our evolution equation reduces to the well-known BJKP equation governing the evolution of four reggeized-gluon state in the dilute limit. To do this, again we first ignore the non-linear terms in the evolution equation, then we expand the Wilson lines and keep terms of the order α^4 in eq. (10). Since the BJKP equation is written in momentum space, we will start by Fourier transforming T_Q (ignoring T at the moment) to momentum space and disregard any contribution which leads to a vanishing external momentum. We define ²

$$T_4(l_1, l_2, l_3, l_4) \equiv \int d^2 r d^2 \bar{r} d^2 \bar{s} d^2 s e^{i(l_1 \cdot r + l_2 \cdot \bar{r} + l_3 \cdot \bar{s} + l_4 \cdot s)} T_4(r, \bar{r}, \bar{s}, s) \quad (14)$$

where l_1, l_2, l_3, l_4 are two-dimensional external transverse momenta satisfying overall transverse momentum conservation so that there are only three independent momenta. This corresponds to having a choice in picking the origin of the coordinate space on the transverse plane. One can then right away see that the last term in (8) convoluted with

² We will use the notation T_4 here to denote the $\sim O(\alpha^4)$ terms in the expansion of T_Q so that $T_4 \equiv \frac{1}{N_c} \text{Tr} [\alpha \alpha \alpha \alpha]$.

$T_4(z, \bar{r}, \bar{s}, s)$ will give a $\delta^2(l_1)$ since it does not depend on coordinate r . A similar argument shows that the last term in each kernel in the first 4 lines in eq. (10) (the "real" terms) will lead to a delta function which sets one of the external momenta to zero. Since the external momenta of the reggeized-gluons are assumed to be finite (non-zero), all these terms can be safely ignored. We now consider the contribution of the "virtual" terms, line 5 in eq. (10). Upon Fourier transforming, we get

$$-8 \frac{N_c \alpha_s}{(2\pi)^2} \int \frac{d^2 p_t}{p_t^2} T_4(l_1, l_2, l_3, l_4) + 4 \frac{N_c \alpha_s}{(2\pi)^2} \int \frac{d^2 p_t}{p_t^2} T_4(p_t + l_1, l_2 - p_t, l_3, l_4) + \dots \quad (15)$$

with a cyclic permutation of the external momenta in the second term understood. The first term is part of the virtual corrections while the second term is part of the real corrections in momentum space. Let us consider now the contribution of "real" terms. Fourier transforming the non-zero terms in the first line of eq. (10) gives

$$2 \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 p_t \left[\frac{p_t \cdot (p_t - l_1)}{p_t^2 (p_t - l_1)^2} T_4(l_1, l_2, l_3, l_4) + 2 \frac{p_t \cdot l_1}{p_t^2 l_1^2} T_4(p_t + l_1, l_2 - p_t, l_3, l_4) \right] \quad (16)$$

The first term in eq. (16) is part of the virtual corrections (in momentum space) while the second term is part of the real corrections. With a slight rearrangement of the first term one can rewrite the contribution of the first line in eq. (10) as

$$2 \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 p_t \left\{ \left[\frac{1}{p_t^2} - \frac{l_1^2}{2 p_t^2 (p_t - l_1)^2} \right] T_4(l_1, l_2, l_3, l_4) + 2 \frac{p_t \cdot l_1}{p_t^2 l_1^2} T_4(p_t + l_1, l_2 - p_t, l_3, l_4) \right\} \quad (17)$$

It is clear that the first term in the square bracket in eq. (17) partially cancels the first term in eq. (15). This cancellation becomes complete when we include the similar contributions from the lines 2 – 4 in eq. (10) so that the only virtual correction left so far is the second term in the square bracket in (17). Including the contribution of the second line to the real part (only the terms which lead to T_4 with the same argument, at the moment) gives

$$\begin{aligned} \frac{d}{dy} T_4(l_1, l_2, l_3, l_4) &= \frac{N_c \alpha_s}{\pi^2} \int d^2 p_t \left[\frac{1}{p_t^2} + \frac{p_t \cdot l_1}{p_t^2 l_1^2} - \frac{p_t \cdot l_2}{p_t^2 l_2^2} - \frac{l_1 \cdot l_2}{l_1^2 l_2^2} \right] T_4(p_t + l_1, l_2 - p_t, l_3, l_4) + \dots \\ &- \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 p_t \left[\frac{l_1^2}{p_t^2 (l_1 - p_t)^2} + \{l_1 \rightarrow l_2, l_3, l_4\} \right] T_4(l_1, l_2, l_3, l_4) \end{aligned} \quad (18)$$

where \dots stands for real contributions obtained by appropriate permutation of the external momenta. Finally we note that the term proportional to $l_1 \cdot l_2$ comes from keeping $O(\sim \alpha^2)$ in the expansion of V_z and setting one of the other V 's to unity, for example, taking $V_{\bar{r}} = 1$ and $\alpha^2(z)$ in the first line of eq. (10). It is clear that the virtual terms in eq. (18) are already in exact agreement with one gets from BJKP equation [14, 15] but the real terms look different. To show agreement of the real terms with the BJKP equation, we rewrite this equation for color charge density ρ rather than the gauge field α (this does not affect the virtual corrections). To this end, we note that the square bracket in the real term in eq. (18) can be rewritten as

$$\left[\frac{1}{p_t^2} + \frac{p_t \cdot l_1}{p_t^2 l_1^2} - \frac{p_t \cdot l_2}{p_t^2 l_2^2} - \frac{l_1 \cdot l_2}{l_1^2 l_2^2} \right] = \frac{1}{2} \left[\frac{(p_t + l_1)^2}{p_t^2 l_1^2} + \frac{(p_t - l_2)^2}{p_t^2 l_2^2} - \frac{(l_1 + l_2)^2}{l_1^2 l_2^2} \right]$$

Recalling the relation between gauge field α and color charge density ρ ,

$$\alpha(p_t) \sim \frac{\rho(p_t)}{p_t^2} \quad (19)$$

and defining $\hat{T}_4(l_1, l_2, l_3, l_4) = \frac{1}{N_c} \text{Tr} \rho(l_1) \rho(l_2) \rho(l_3) \rho(l_4)$, we multiply both sides of eq. (18) with $l_1^2 l_2^2 l_3^2 l_4^2$ which effectively removes the external legs. Eq. (18) can then be written as

$$\begin{aligned} \frac{d}{dy} \hat{T}_4(l_1, l_2, l_3, l_4) &= \frac{N_c \alpha_s}{\pi^2} \int d^2 p_t \left[\frac{p^i}{p_t^2} - \frac{(p^i - l_1^i)}{(p_t + l_1)^2} \right] \cdot \left[\frac{p^i}{p_t^2} - \frac{(p^i - l_2^i)}{(p_t + l_2)^2} \right] \hat{T}_4(p_t + l_1, l_2 - p_t, l_3, l_4) + \dots \\ &- \frac{N_c \alpha_s}{(2\pi)^2} \int d^2 p_t \left[\frac{l_1^2}{p_t^2 (l_1 - p_t)^2} + \{l_1 \rightarrow l_2, l_3, l_4\} \right] \hat{T}_4(l_1, l_2, l_3, l_4) \end{aligned} \quad (20)$$

This is our final result and corresponds to the evolution of \hat{T}_4 after one step in rapidity y as depicted (the real part) in Fig. (1). We have checked that it agrees with the expressions given in [14, 15].

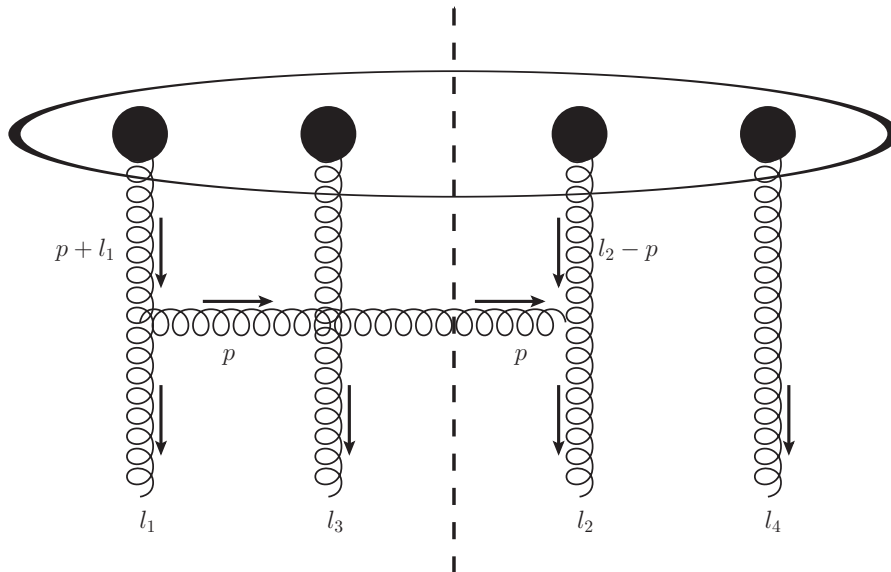


FIG. 1: Evolution of the four-point function \hat{T}_4 after one step in rapidity as given by eq. (20). Shown is one of the real diagrams only and the dashed line represents a cut.

There are several points that need to be clarified; first, we have completely disregarded the dipole terms ($\sim T$) here even though they also contain $O(\alpha^4)$ terms. Since $T(r, s)$ depends only on two external transverse coordinates r, s , $O(\alpha^4)$ terms will necessarily involve two pairs of gauge fields at the same point. Assuming rotational invariance on the transverse coordinate plane, this leads to setting two of the external momenta equal to each other which takes one back to the BFKL ladders. Therefore, these terms are not relevant for our purpose. A second point is the color averaging denoted by $\langle \dots \rangle$. We have not made any assumptions about the color averaging [20] and the evolution equation derived is independent of how one performs this averaging. Furthermore, the overall color structure of the equation seems to be more general than the BJKP equation since here one has a trace of four color matrices in the fundamental representation on both sides of the equation. This trace could be written in terms of products of the group structure constants $\delta^{ab}, f^{abc}, d^{abc}$ whereas the BJKP equation is for the exchange of four reggeized-gluon state in a symmetric color singlet state. One expects that $\delta\delta$ terms would lead to a topology which is equivalent to exchange of two independent BFKL pomerons which would then be disregarded. Therefore, one would only consider the color symmetric structures involving d 's.

In summary, we have shown in this preliminary study that the JIMWLK evolution equation for the quadrupole operator can be reduced to the BJKP equation for the real part of the four reggeized-gluon exchange amplitude. To do this, we first ignore the non-linear (recombination) terms in quadrupole evolution equation, and then expand the Wilson lines in terms of the gauge field (or equivalently, the color charge density). This approximation should be valid when the external momenta are larger than the saturation scale, i.e., in the dilute region. The quadrupole evolution equation reduces to a sum of independent BFKL equations in $O(\rho^2)$ and to the BJKP equation when one looks at the terms of order $\sim \rho^4$. This suggests that the JIMWLK evolution equation for the n -pole operator $\frac{1}{N_c} \langle \text{Tr } V(x_1) V^\dagger(x_2) \dots V^\dagger(x_n) \rangle$ in the linear limit (dilute region) may be equivalent to the BJKP hierarchy for the imaginary part of the n reggeized-gluon exchange amplitude. This would be very useful since there is much that is known about the BJKP equation and its properties but not much is known about the properties of the JIMWLK equation in analytic form. Proving the equivalence between linearized JIMWLK and BJKP equations may not be so difficult since the JIMWLK evolution equation for $\frac{1}{N_c} \langle \text{Tr } V(x_1) V^\dagger(x_2) \dots V^\dagger(x_n) \rangle$ can almost be written down by inspection in analogy with the pattern seen in eq. (5). The problem reduces to keeping track of which quark line radiates a gluon and counting all the possibilities since all emission kernels are just the standard dipole kernel. It would also be interesting to investigate the connection between the non-linear terms in the JIMWLK equation and multi-pomeron vertices employed in reggeized-gluon approach to high energy scattering. These issues are beyond the scope of this preliminary work and will be reported elsewhere.

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